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Spectral Analysis of Fractional Kinetic Equations with Random Data

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We present a spectral representation of the mean-square solution of the fractional kinetic equation (also known as fractional diffusion equation) with random initial condition. Gaussian and non-Gaussian limiting distributions of the renormalized solution of the fractional-in-time and in-space kinetic equation are described in terms of multiple stochastic integral representations.

KEY WORDS: Fractional kinetic equation; fractional diffusion equation; scaling laws; renormalised solution; long-range dependence; non-Gaussian scenario; Mittag–Leffler function; Bessel potential; Riesz potential; stable distributions.

1. INTRODUCTION

Fractional diffusion equations/fractional kinetic equations were introduced to describe physical phenomena such as diffusion in porous media with fractal geometry, kinematics in viscoelastic media, relaxation processes in complex systems (including viscoelastic materials, glassy materials, synthetic polymers, biopolymers), propagation of seismic waves, anomalous diffusion and turbulence (see Caputo,[16] Glöckle and Nonnenmacher,[34] Mainardi,[49, 51] Saichev and Zaslavski,[67] Zaslavski,[80] Mainardi and Tomirotti,[52] Kobelev *et al.*,[41] Metzler *et al.*,[54] Hilfer,[37] and the references therein). These equations are obtained from the classical diffusion

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equation by replacing the first and/or second-order derivative by a fractional derivative (see Oldham and Spanier,[58] Samko *et al.*,[68] Miller and Ross,[55] Gorenflo and Mainardi,[36] Djrbashian,[21] Podlubny,[60] Butzer and Westphal[15] for different types of fractional derivatives, fractional integrals or fractional operators in general and their properties). In the non-stochastic situation, fractional diffusion equations/fractional kinetic equations have been studied by Schneider and Wyss,[71] Kochubei,[42,43] Fujita,[30] Prüss,[62] Mainardi,[50] Saichev and Zaslavski,[67] Zaslavski,[80] Gorenflo *et al.*,[35] and others. A more general fractional Burgers equation has been considered by Biler *et al.*[11] (see also Woyczynski[79]).

We are interested in fractional-in-time and in-space diffusion equations with random initial conditions as models of random fields with singular spectra which describe the singular and fractal properties of real data arising in applied fields such as turbulence, hydrology, ecology, geophysics, air pollution, economics and finance. It will be seen that renormalized solutions of fractional diffusion equations with random data may possess long-range dependence (LRD) and intermittency.

The fractional operators are natural mathematical objects to describe the LRD and/or intermittency phenomena. In particular, Gay and Heyde[32] introduced a class of random fields that allow LRD via the stochastic operational Laplace equation with fractional Laplace operator. Anh *et al.*,[5] Angulo *et al.*[4] introduced a fractional stochastic heat equation in which the n-dimensional Laplacian Δ is replaced by a fractional Laplacian of the form $-(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}$, $\alpha>0$, $\gamma\geqslant 0$, where the operators $-(I-\Delta)^{\gamma/2}$, $\gamma\geqslant 0$ and $(-\Delta)^{\alpha/2}$, $\alpha>0$, are interpreted as inverses of the Bessel and Riesz potentials respectively. In fact, based on a new concept of duality of generalised random fields defined on fractional Sobolev spaces introduced in Ruiz-Medina *et al.*,[64] Anh *et al.*[5] prove the existence of a class of random fields defined by the equation

$$(I-\Delta)^{\frac{\gamma}{2}}(-\Delta)^{\frac{\alpha}{2}}c(x)=e(x), \qquad e(x)\equiv \text{white noise}, \; x\in\mathbb{R}^n, \tag{1.1}$$

or equivalently (in the sense of second-order moments) by the spectral density

$$f(\lambda)=\frac{c}{|\lambda|^{2\alpha}}\frac{1}{(1+|\lambda|^2)^{\gamma}}, \qquad c>0, \qquad \frac{n}{2}<\alpha<1+\frac{n}{2}, \qquad \gamma\geqslant 0, \quad \lambda\in\mathbb{R}^n. \tag{1.2}$$

These random fields were named fractional Riesz–Bessel motion (fRBm). It is noted that in view of (1.2), fractional Brownian motion (fBm) is a limiting case of fRBm with $\gamma=0$, and fRBm displays LRD (as $|\lambda|\to 0$) of order

α for $\alpha > \frac{1+n}{2}$. It displays second-order intermittency, i.e, clustering of extreme values, (as $|\lambda| \to \infty$) of order $\alpha + \gamma$. The presence of the Bessel operator is essential for a study of stationary solutions of (1.1). In fact, this case requires $0 < \alpha < \frac{n}{2}$ and $\alpha + \gamma > \frac{n}{2}$; that is, the condition $\gamma > 0$ is necessary. The parameter of the Bessel operator is also useful in determining suitable conditions for the spectral density of the solutions of fractional kinetic equations to belong to $L_1(\mathbb{R}^n)$ (see (3.2) below). The importance of this parameter is discussed after (3.5) below.

On the other hand, random fields with singular spectra can be obtained as rescaled solutions of the linear diffusion equations with singular initial conditions (see Albeverio *et al.*,[1] Leonenko and Woyczynski,[46] Anh and Leonenko[6–8]). Recently, several researchers investigated the Burgers equation with random data which relates to the heat equation via the Cole–Hopf transformation (see Bulinski and Molchanov,[14] Albeverio *et al.*,[1] Funaki *et al.*,[31] Leonenko and Woyczynski,[47] Woyczynski,[79] Leonenko,[45] Bertoin,[10] Ryan,[66] Dermoune *et al.*[20]). Beghin *et al.*[9] considered scaling laws for linear Korteweg–de Vries equation or Airy equation with random data. Anh and Leonenko[8, 7] presented the theory of renormalization and homogenization of fractional-in-time or in-space diffusion equation with random data.

Our paper is motivated by the works of Leonenko and Woyczynski,[46] Ruiz-Medina *et al.*,[65] Anh and Leonenko[6–8] in which Gaussian and non-Gaussian scenarios are presented for the classical heat equation (Leonenko and Woyczynski,[46] Anh and Leonenko[6]), fractional-in-time diffusion-wave equation (Anh and Leonenko[8, 7]) and fractional-in-space diffusion equation (Ruiz-Medina *et al.*,[65] Anh and Leonenko[7]) with singular and possibly non-Gaussian initial conditions.

We generalize the results of Anh and Leonenko[6–8] and Ruiz-Medina *et al.*[65] to the fractional-in-time and fractional-in-space diffusion equation and obtain new Gaussian and non-Gaussian scenarios for the renormalized solution of the resulting fractional diffusion equation with random data. In a sense, our results are the non-Gaussian central limit theorems for solutions of generalized kinetic equations with singular data (see Taqqu,[75] Dobrushin and Major[24] and others). Note that the renormalization, normalizing factors, Gaussian and non-Gaussian limiting fields obtained in this paper are new or at least in a more general form. The Green functions of generalized kinetic equations as well as the corresponding spectral representation are also new.

The paper is organized as follows. Sections 2 and 3 provide preliminaries on the fractional-in-space and in-time diffusion equations with random initial conditions, including some new results concerning the Green functions of these equations. Section 4 describes the main results of this paper including spectral representation, Gaussian and non-Gaussian scaling laws. The proof of these results are given in Section 5.

2. FRACTIONAL KINETIC EQUATION

We consider the following fractional kinetic equation/fractional diffusion equation

$$\frac{\partial^\beta u}{\partial t^\beta} = -\mu(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}u, \qquad \mu > 0 \tag{2.1}$$

subject to the initial condition

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{2.2}$$

where $u = u(t, x)$, $0 < t \leqslant T$, $x \in \mathbb{R}^n$, is the kinetic field and $\beta \in (0, 1]$, $\gamma \geqslant 0$, $\alpha > 0$ are fractional parameters. We shall concentrate on the case of random initial condition, that is,

$$u_0(x) = \eta(x), \qquad x \in \mathbb{R}^n, \tag{2.3}$$

where $\eta(x) = \eta(\omega, x)$, $\omega \in \Omega$, $x \in \mathbb{R}^n$, is a measurable random field defined on a suitable complete probability space $(\Omega, \mathscr{A}, P)$. Here, Δ is the n-dimensional Laplace operator, and the operators $-(I-\Delta)^{\gamma/2}$, $\gamma \geqslant 0$, and $(-\Delta)^{\alpha/2}$, $\alpha > 0$, are interpreted as inverses of the Bessel and Riesz potentials respectively (see Appendix B). Both Bessel and Riesz potentials are considered to be defined in a weak sense, in frequency domain, in terms of fractional Sobolev spaces.

The time derivative of order $\beta \in (0, 1]$ is defined as follows:

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \dfrac{\partial u}{\partial t}(t, x), & \text{if } \beta = 1, \\ (\mathscr{D}_t^\beta u)(t, x), & \text{if } \beta \in (0, 1), \end{cases} \tag{2.4}$$

where

$$(\mathscr{D}_t^\beta u)(t, x) = \frac{1}{\Gamma(1-\beta)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\beta}\, u(\tau, x)\, d\tau - \frac{u(0, x)}{t^\beta}\right], \qquad 0 < t \leqslant T,$$

is the regularized fractional derivative or fractional derivative in the Caputo–Djrbashian sense (see Caputo,[16] Djrbashian and Nersesian,[22] Kochubei,[42] Djrbashian,[21] Podlubny,[60] or Appendix A). The idea of regularization can be found in Caputo,[16, 17] Caputo and Mainardi,[18] Gorenflo and Mainardi.[36] Note that for $\gamma = 0$, $\beta = 1$, $\alpha = 2$, Eq. (2.1) is the classical linear diffusion equation or heat equation with random initial condition (see Rosenblatt,[63] Anh and Leonenko,[6] and others).

In the non-stochastic situation the fractional-in-time diffusion equation, which formally corresponds to Eq. (2.1) with $\gamma = 0$, $\alpha = 2$, has been studied by many authors. In particular, Schneider and Wyss,[71] Schneider,[70] Fujita,[30] Prüss,[62] Engler[26] considered the fractional integro-differential equations or Volterra-type equations while Kochubei,[42] Hilfer,[37] Kostin[44] investigated a Cauchy problem for fractional evolution equations in Banach space with fractional derivatives. Mainardi[49–51] used the fractional derivatives in the Caputo sense to solve the Cauchy problem for the one-dimensional fractional-in-time diffusion equation (see also Gorenflo and Mainardi[51]). The fractional-in-space diffusion equation, which formally corresponds to Eq. (2.1) with $\beta = 1$, $\gamma = 0$, $0 < \alpha \leqslant 2$, was first considered by Feller[29] and then many others (see Stroock[74] for example) in the context of fractional diffusion leading to a study of Markov processes or Lévy processes governed by stable distributions (see also Uchaikin and Zolotarev[77]). Saichev and Zaslavski,[67] Kobelev *et al.*,[41] Metzler *et al.*,[54] Zaslavski,[80] Gorenflo *et al.*[35] proposed the one-dimensional fractional generalization of the diffusion equation incorporating fractional derivatives with respect to time and space coordinates. It was introduced to describe anomalous kinetics of simple dynamical systems with chaotic motion. Hilfer[37] introduced a two-parameter fractional-in-time diffusion equation.

Hochberg and Orsingher,[38] Beghin *et al.*[9] considered higher-order parabolic equations, which formally correspond to Eq. (2.1) with $\beta = 1$, $\gamma = 0$, $n = 1$ and $\alpha = 3, 4, \ldots$ They presented solutions of such equations in the form of density functions of iterated Brownian motion (Hochberg and Orsingher[38]) or Gaussian limiting behaviour of the rescaled solution of the linear Korteweg–de Vries equation with random data (Beghin *et al.*[9]).

We shall extensively use the following entire function of order $1/\beta$ and type 1:

$$E_\beta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta j + 1)}, \qquad z \in \mathbb{C}^1, \quad \beta > 0.$$

This function is known as the Mittag–Leffler function (see Erdély *et al.*,[27] pp. 206–212, or Djrbashian[21]). In particular, for real $x \geqslant 0$, $\beta > 0$,

$$E_\beta(-x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{\Gamma(\beta j + 1)} \tag{2.5}$$

is infinitely differentiable and completely monotonic if $0<\beta<1$, that is,

$$(-1)^k \frac{d^k}{dx^k} E_\beta(-x) \geqslant 0, \qquad x \geqslant 0, \quad 0<\beta<1, \quad k=0, 1, 2, \ldots$$

For real $x \geqslant 0$ and $\beta<1$,

$$E_\beta(-x) = \frac{\sin(\pi\beta)}{\pi\beta} \int_0^\infty \frac{\exp\{-(xt)^{1/\beta}\}\, dt}{t^2+2t\cos(\pi\beta)+1}. \tag{2.6}$$

In particular, for $x \geqslant 0$,

$$E_1(-x) = e^{-x}, \qquad E_{1/2}(-x) = e^{x^2}\left(1-\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt\right).$$

From (C.5) and (C.12) in Appendix C, we obtain the following asymptotic expansion:

$$E_\beta(-x) = -\sum_{k=1}^{N} \frac{(-1)^k x^{-k}}{\Gamma(1-\beta k)} + O(|x|^{-N-1}) \tag{2.7}$$

as $x \to \infty$, where $\beta<1$ (see also Djrbashian,[21] p. 5). We shall recall some important results on the Mittag–Leffler function (see Djrbashian and Nersesian,[22] Theorems 5 and 6, Kochubei,[43] Podlubny,[60] or Engler,[26] Appendix) in the following

Theorem 1. The function

$$u(t) = E_\beta(-at^\beta)$$

is the unique solution in $L_p(0, T)$, $p \geqslant 1$, of the Cauchy problem for the ordinary fractional equation

$$\mathscr{D}^\beta u(t) + au(t) = 0, \qquad u(0)=1, \quad a>0, \tag{2.8}$$

where

$$\mathscr{D}^\beta u(t) = (\mathscr{D}^\beta u)(t) = (\mathscr{R}^\beta u)(t) - \frac{u(0)}{\Gamma(1-\beta)\, t^\beta}$$

and

$$(\mathscr{R}^{\beta}u)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{u(\tau)\, d\tau}{(t-\tau)^{\beta}}$$

is the Riemann–Liouville fractional derivative (see Appendix A).

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing $C^{\infty}(\mathbb{R}^n)$-functions with the dual $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$, which is the space of tempered distributions (see, for example, Dautray and Lions,[19] Vol. 2, Appendix). Consider now Eq. (2.1) subject to the initial condition

$$u_0(x) = \delta(x) \in \mathscr{S}', \qquad x \in \mathbb{R}^n, \tag{2.9}$$

where $\delta(x)$ is the Dirac delta-function. We shall denote by $\hat{u} = \mathscr{F}_x[u]$ the Fourier transform of a distribution $u \in \mathscr{S}'$ with respect to the space variable $x \in \mathbb{R}^n$. In particular, let $\hat{G} = \hat{G}(t, \xi)$, $t > 0$, $\xi \in \mathbb{R}^n$ being the dual variable of $x \in \mathbb{R}^n$, be the Fourier transform of the fundamental solution (i.e., the Green function) of the Cauchy problem (2.1) and (2.9). By the Fourier transform with respect to x, (2.1) and (2.9) is equivalent (see Appendix B) to the Cauchy problem

$$(\mathscr{D}_t^{\beta}\hat{G})(t, \xi) = -\mu\, |\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2}\, \hat{G}, \qquad \hat{G}(0, \xi) = 1. \tag{2.10}$$

To solve (2.10) we associate with it an ordinary fractional differential equation (2.8) depending on the parameter

$$a = \mu\, |\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2} \geqslant 0.$$

Using standard arguments (see, for example, Dautray and Lions,[19] Vol. 5, pp. 8–15) and Theorem 1 we arrive at the following

Theorem 2. The Cauchy problem (2.10) has a unique solution given by

$$\hat{G}(t, \xi) = E_{\beta}(-\mu t^{\beta}\, |\xi|^{\alpha}\, (1+|\xi|^2)^{\gamma/2}), \tag{2.11}$$

where $E_{\beta}(-x)$, $x \geqslant 0$, is the Mittag–Leffler function (2.5) of the negative real argument.

Moreover, for $u_0(x) \in \mathscr{S}$ (or $u_0(x) \in \mathscr{S}'$ and has a compact support) the initial value problem (2.1) and (2.9) has the unique solution

$$u(t, x) = \int_{\mathbb{R}^n} G(t, x-y)\, u_0(y)\, dy, \tag{2.12}$$

where the Green function $G(t, x)$, $0 < t \leqslant T$, $x \in \mathbb{R}^n$, satisfies (2.11).

In the next section, we present some explicit expressions for the fundamental solution G of the initial-value problem (2.1) and (2.9) in terms of H-functions (see Appendix C). For certain special values of the fractional parameters α, β and γ, the known results of Schneider and Wyss[71] and Mainardi[50] are recovered. For a fuller collection of explicit solutions of pseudo-differential equations in terms of H-functions, see Hilfer[37].

3. THE GREEN FUNCTION

Let $G(t, x)$, $0 < t \leqslant T$, $x \in \mathbb{R}^n$, be the Green function of the fractional kinetic equation (2.1) whose Fourier transform is given by (2.11). The inverse Fourier transform can be written as

$$G(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} E_\beta(-\mu t^\beta |\lambda|^\alpha (1+|\lambda|^2)^{\gamma/2})\, d\lambda. \tag{3.1}$$

For $\alpha > 0$, $\beta \in (0, 1]$ and $\gamma \geqslant 0$ such that

$$E_\beta(-\mu t^\beta |\lambda|^\alpha (1+|\lambda|^2)^{\gamma/2}) \in L_1(\mathbb{R}^n), \tag{3.2}$$

the Green function

$$G(t, x) \in L_1(\mathbb{R}^n). \tag{3.3}$$

Clearly, the inverse Fourier transform (3.1) is rotation invariant (in λ) and the function (2.11) is rotation invariant in ξ. Thus, the inverse Fourier transform (3.1) can be represented by the Hankel method as

$$G(t, x) = \frac{(2\pi)^{-n/2}}{|x|^{(n-2)/2}} \int_0^\infty \rho^{n/2} \mathscr{J}_{(n-2)/2}(|x|\, \rho)\, E_\beta(-\rho^\alpha (1+\rho^2)^{\gamma/2}\, t^\beta \mu)\, d\rho, \tag{3.4}$$

where $\mathscr{J}_\nu(z)$ is the Bessel function of the first kind of order ν (see (C.14) in Appendix C). The Hankel transform (3.4) exists (see, for example, Theorem 1 of Dautray and Lions,[19] Vol. 2, p. 48) for $\gamma \geqslant 0$, $\beta \in (0, 1]$, $\alpha > 0$ such that

$$\rho^{(n/2)-(1/2)} E_\beta(-\rho^\alpha (1+\rho^2)^{\gamma/2}\, t^\beta \mu) \in L_1(0, \infty). \tag{3.5}$$

From (2.7) we obtain that for a fixed $t \in (0, T]$ the condition (3.2) holds for every $\beta \in (0, 1]$ if $\alpha + \gamma > n$, while the condition (3.5) is satisfied for $\alpha + \gamma > (n+1)/2$. From these ranges we see the important role of the parameter γ in Eq. (2.1).

Remark 1. In fact, (3.1) and (3.4) hold for broader ranges of α, β and γ if we are able to prove that $G(t, \cdot) \in L_1(\mathbb{R}^n)$, in which case, we may compute the Fourier transform of $G(t, \cdot)$ by the Hankel method via the same integral formula. In this case, no problem arises. Otherwise, we may interpret the Hankel transform (3.4) as an $L_2((0, \infty), \rho d\rho)$ isometry for $\alpha+\gamma > n/2$. An analysis of (3.1) or (3.4) is possible also within the framework of Schwartz distributions.

Remark 2. The Green function (3.1) or (3.4) of the fractional diffusion equation (2.1) provides the most general form to our knowledge. For the special values $\beta = 1$ and/or $\alpha = 2$ the known results can be recovered.

Let us consider the initial-value problem (2.1) and (2.9) with $\beta = 1$ (fractional-in-space kinetic equation with factorization of the Laplacian). In this case the Mittag–Leffler function $E_1(-x) = e^{-x}$, $x \geqslant 0$, and from (3.1) we obtain an explicit expression of the Green function (see Anh and Leonenko[7]) as

$$G(t, x) = p(t, x; \alpha, \gamma, \mu)$$
$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \exp\{i\langle\lambda, x\rangle - \mu t\, |\lambda|^{\alpha}\, (1+|\lambda|^2)^{\gamma/2}\}\, d\lambda, \qquad \alpha > 0,\ \gamma \geqslant 0. \tag{3.6}$$

For $\gamma = 0$, $\alpha = 2$ the Green function (3.6) reduces to the n-dimensional isotropic Gaussian density

$$p(t, x; 0, 2, \mu) = G(t, x) = (4\pi\mu t)^{-n/2} \exp\left\{ -\frac{|x|^2}{4\mu t} \right\}. \tag{3.7}$$

For $\gamma = 0$, $\alpha = 1$,

$$p(t, x; 0, 1, \mu) = \Gamma\left(\frac{n+1}{2}\right) \pi^{-(n+1)/2} \mu t [(\mu t)^2 + |x|^2]^{-(n+1)/2}$$

is the density function of the n-dimensional symmetric Cauchy distribution, while for $\gamma = 0$, $\alpha \in (0, 2]$,

$$p\left(\frac{t}{2}, x; 0, \alpha, 1\right) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle\lambda, x\rangle - (t/2)\, |\lambda|^{\alpha}} d\lambda \tag{3.8}$$

is the density function of the $n-$dimensional symmetric stable distribution. For a general discussion of stable distributions and processes, see

Samorodnitsky and Taqqu,[69] Uchaikin and Zolotarev.[77] Note that for $\alpha > 2$ the function (3.8) may become negative for some value of x.

It is known (see, for example, Andrews *et al.*,[3] p. 221) that

$$\int_0^\infty \mathcal{J}_\nu(at)\, t^{\nu+1} e^{-p^2t^2} dt = \frac{a^\nu}{(2p^2)^{\nu+1}}\, e^{-a^2/4p^2}, \qquad \operatorname{Re} \nu > -1. \tag{3.9}$$

Combining (2.6), (3.4) and (3.9) we get the following elegant expression for the Green function:

$$G(t, x) = \frac{\sin(\pi\beta)}{2\pi^{3/2}\beta\mu^{1/2\beta}t^{1/2}} \int_0^\infty \frac{\exp\{-|x|^2/4t\mu^{1/\beta}u^{1/\beta}\}\, du}{u^{1/2\beta}(u^2+2u\cos(\pi\beta)+1)},$$

which holds for $n=1$, $0<\beta<1$, $\alpha=2\beta\in(0,2)$, $\gamma=0$. For $\gamma=0$, we are able to give a new explicit expression for the Green function (3.4) in terms of H-functions (see Appendix C). Applying (C.18) of Appendix C to (3.4) yields

$$\begin{aligned} G(t, x) &= \frac{\pi^{-n/2}}{|x|^n} H^{2,1}_{2,3}\left(\frac{|x|^\alpha}{2^\alpha t^\beta \mu} \middle| \begin{matrix} (1,1) & (1,\beta) & \\ (n/2, \alpha/2) & (1,1) & (1,\alpha/2) \end{matrix}\right) \\ &= \frac{\pi^{-n/2}}{|x|^n} H^{1,2}_{3,2}\left(\frac{2^\alpha t^\beta \mu}{|x|^\alpha} \middle| \begin{matrix} (1-(n/2), \alpha/2) & (0,1) & (0,\alpha/2) \\ (0,1) & (0,\beta) & \end{matrix}\right), \end{aligned} \tag{3.10}$$

where $H^{2,1}_{2,3}$ and $H^{1,2}_{3,2}$ are H-functions defined by (C.1) or (C.3) of Appendix C. From (C.19) and (C.2a–C.2h)) we obtain that (3.10) holds at least for

$$\beta \in (0, 1], \ \min(n, 2, \alpha) > (n-1)/2, \qquad |x| \neq 0. \tag{3.11}$$

Remark 3. From the equation (10.1.1) of Srivastava *et al.*[72] we obtain that for $n=1$ the function $q(u)=\alpha G(t,u)$, $u>0$, is a density function for $\beta\in(0,2]$, $0<\alpha\leqslant 2$, $\beta<\alpha$. It means that in this region $G(t,x)\in L_1(\mathbb{R}^1)$ and $G(t,x)\geqslant 0$ for all $n\geqslant 1$.

If we apply the relation (C.5) in Appendix C with $m=1$, $n=2$, $p=3$, $q=2$, $s_k=-k$, $k=0,1,2,\ldots$, $B(s)=\Gamma((n-\alpha s)/2)\,\Gamma(1-s)$, $C(s)=\Gamma(1-\beta s)$, $D(s)=\Gamma(\alpha s/2)$ to (3.10) we get the following series representation:

$$G(t, x) = \frac{\pi^{-n/2}}{|x|^n} \sum_{k=0}^\infty (-1)^k \left(\frac{|x|^\alpha}{2^\alpha t^\beta \mu}\right)^{-k} \frac{\Gamma((n+\alpha k)/2)}{\Gamma(1+\beta k)\,\Gamma(-\alpha k/2)}, \qquad \beta > \alpha. \tag{3.12}$$

It should be noted that formula (3.12) was first obtained by Gorenflo *et al.*[35] for the case $n = 1$ by a different method.

In the case $\beta < \alpha$ we obtain from (C.6) the following representation:

$$G(t, x) = \frac{\pi^{-n/2}}{|x|^n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{|x|^\alpha}{2^\alpha t^\beta}\right)^{k+1} \frac{\Gamma(n-\alpha-\alpha k/2)}{\Gamma(1-\beta-\beta k)\,\Gamma(\alpha+\alpha k/2)} \tag{3.13}$$

This formula again was obtained by Gorenflo *et al.*[35] in the case $n = 1$. Moreover, for $n = 1$, Gorenflo *et al.*[35] note that if $\beta = \alpha$ the series representation is given by (3.13) if $0 < t < |x|$, and by (3.12) if $0 < |x| < t$. Gorenflo *et al.*[35] found that if $\beta = \alpha$, $n = 1$ both formulae (3.13) and (3.12) can be simplified to a fractional Cauchy kernel:

$$G(t, x) = \frac{1}{\pi} \frac{|x|^{\alpha-1}\, t^\alpha \sin\dfrac{\alpha\pi}{2}}{t^{2\alpha} + 2\,|x|^\alpha\, t^\alpha \cos\dfrac{\alpha\pi}{2} + |x|^{2\alpha}}.$$

It should be noted that for $\beta = \alpha = 1$, $n = 1$ the above kernel becomes the well-known Cauchy kernel:

$$G(t, x) = \frac{1}{\pi} \frac{t}{t^2 + x^2}.$$

We shall see below that for the special values of $\beta = 1$ and/or $\alpha = 2$ the known results can be recovered. For example, if $\gamma = 0$, $\alpha = 2$, $\beta \in (0, 1]$, the Green function (3.10) is reduced to

$$G(t, x) = \frac{\pi^{-n/2}}{|x|^n} H^{2,1}_{2,3}\left(\frac{|x|^2}{4t^\beta \mu} \,\middle|\, \begin{matrix} (1, 1) & (1, \beta) & \\ (n/2, 1) & (1, 1) & (1, 1) \end{matrix}\right). \tag{3.14}$$

Applying (C.7) to (3.14) yields

$$G(t, x) = \frac{\pi^{-n/2}}{|x|^n} H^{2,0}_{1,2}\left(\frac{|x|^2}{4t^\beta \mu} \,\middle|\, \begin{matrix} (1, \beta) & \\ (n/2, 1) & (1, 1) \end{matrix}\right). \tag{3.15}$$

Applying (C.9) with $c = 1/\beta$ to (3.14) we get, for $\mu = 1$,

$$G(t, x) = \frac{\pi^{-n/2}}{\beta |x|^n} H^{2,0}_{1,2}\left(\frac{|x|^{2/\beta}}{2^{2/\beta} t} \,\middle|\, \begin{matrix} (1, 1) & \\ (n/2, 1/\beta) & (1, 1/\beta) \end{matrix}\right). \tag{3.16}$$

The Green function (3.16) is exactly the Green function (3.4) of Schneider and Wyss (see ref. 71), while the formula (3.15) is the Green function of

Kochubei ([42]). For $\beta = 1$, (3.14), (3.15) and (3.16) reduce to (3.7) by using (C.10). Moreover, applying (C.11) with $\sigma = -n/2$ to (3.15) we get

$$G(t, x) = (4\pi t^\beta)^{-n/2} H^{2,0}_{1,2}\left(\frac{|x|^2}{4t^2} \middle| \begin{matrix} (1-(\beta n)/2, \beta) & \\ (0, 1) & (1-(n/2), 1) \end{matrix} \right). \tag{3.17}$$

The Green functions (3.15), (3.16) and (3.17) have exponential behaviour as $|x| \to \infty$ according to (C.8) and $G(t, \cdot) \in L_1(\mathbb{R}^n)$. This is a probability density in $\mathbb{R}^n$ for every $\beta \in (0, 1]$. Hilfer[37] considered a two-parameter fractional-in-time diffusion equation and obtained a fundamental solution in terms of H-functions.

The Green function (3.17) coincides with the Green function (1.22) of Schneider.[70] This Green function can be written as

$$\begin{aligned} G(t, x) &= (4\pi t^\beta)^{-n/2} g_\beta(|x|^2 t^{-\beta} \tfrac{1}{4}) \\ &= \int_0^\infty G_1(ut^\beta, x) \zeta_\beta(u) \, du, \end{aligned} \tag{3.18}$$

where $G_1(t, x)$ is the Green function (3.7) of the ordinary diffusion equation

$$\begin{aligned} g_\beta(u) &= \int_0^\infty \zeta_\beta(s) e^{-u/s} s^{-n/2} ds, \qquad u > 0, \\ \zeta_\beta(u) &= \frac{1}{\beta} u^{-1-(1/\beta)} \rho_\beta(u^{-1/\beta}) = H^{1,0}_{1,1}\left(u \middle| \begin{matrix} (1-\beta, \beta) \\ (0, 1) \end{matrix} \right) \end{aligned} \tag{3.19}$$

and $\rho_\beta(u)$ is the one-sided stable probability density with Laplace transform (see (C.12))

$$\int_0^\infty e^{-pt} \rho_\beta(t) \, dt = \exp\{-p^\beta\}, \qquad \operatorname{Re} p > 0.$$

Remark 4. The function (3.19) itself is a probability density with Laplace transform $E_\beta(-t), t \geqslant 0$ (see Schneider,[70] for example). The behaviour of $g_\beta(u)$ for small u can be obtained from (C.4) or (C.5) leading to $g_\beta(u) \sim [\Gamma(1/2)/\Gamma(1-(\beta/2))] u^0$, $n = 1$; $g_\beta(u) \sim [-1/\Gamma(1-\beta)] \log u$, $n = 2$; $g_\beta(u) \sim [\Gamma(n/2-1)/\Gamma(1-\beta)] u^{1-(n/2)}$, $n \geqslant 3$. Its asymptotic behaviour for large $u > 0$ is determined by (C.6) and reads

$$g_\beta(u) \sim C u^{-\sigma} \exp\{-c u^\tau\} \qquad \text{with} \quad C = (2-\beta)^{-1/2} \beta^\nu,$$

where $\sigma = n(1-\beta)/[2(2-\beta)]$, $\tau = (2-\beta)^{-1}$, $\nu = [\beta(n+2)-2]/(2(2-\beta))$.

Remark 5. Kochubei[42] presented the singular properties of the fundamental solution (3.15) with estimates both in $t \in (0, T]$ and $x \in \mathbb{R}^n$. Let $c_i > 0$ be positive constants. Then for $t^{-\beta} |x|^2 < 1, x \neq 0$, we have the following estimates: $|G(t, x)| \leqslant c_1 t^{-\beta/2}$, $n = 1$; $|G(t, x)| \leqslant c_2 t^{-\beta}[1 + \ln (t^{-\beta} |x|^2)]$, $n = 2$; $|G(t, x)| \leqslant c_3 t^{-\beta} |x|^{-n+2}$, $n \geqslant 3$. For $t^{-\beta} |x|^2 \geqslant 1$, $|G(t, x)| \leqslant c_4 t^{-\beta n/2} \exp \{-c_5 t^{-\beta/(2-\beta)} |x|^{2/(2-\beta)}\}$. Kochubei[42] proved that under some conditions (see (1)–(3) below) the function

$$u(t, x) = \int_{\mathbb{R}^n} G(t, x-y)\, u_0(y)\, dy \tag{3.20}$$

is the classical solution of the initial-value problem (2.1) and (2.9) with $\alpha = 0$, $\gamma = 2$, $\beta \in (0, 1]$, that is,

(i) $u(t, \cdot) \in C^2(\mathbb{R}^n)$;

(ii) $u(\cdot, x) \in C(0, T)$;

(iii) there exists a fractional integral $(I^{1-\beta}u)(t, x) \in C^1(0, T)$, where

$$(I^{1-\beta}u)(t, x) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta}\, u(x, \tau)\, d\tau;$$

(iv) the function (3.20) satisfies (2.1) and (2.9).

Moreover

$$|u(t, x)| \leqslant c_6 \exp \{h |x|^{2/(2-\beta)}\}, \qquad 0 < h < \mu_0 T^{-\beta/(2-\beta)},$$
$$\mu_0 = (2-\beta)\, \beta^{\beta/(2-\beta)} 2^{-\beta/(2-\beta)};$$

$u(t, x) \in H_\nu^{\beta+\lambda}(0, T)$ if $0 < \lambda + \beta < 1$, $\beta + \lambda < \nu$, where $H_\nu^{\beta+\lambda}(0, T)$ is the class of functions $f(t)$, $t \in (0, T)$ such that $t^\nu f(t)$ satisfies the Hölder condition of order $\beta + \lambda$. The conditions for Kochubei's theorem are the following:

(1) $u_0(x)$, $x \in \mathbb{R}^n$ is continuous;

(2) $u_0(x) \leqslant c_7 \exp \{h |x|^{2/(2-\beta)}\}$, $0 < h < \mu_0 T^{-\beta/(2-\beta)}$, and

(3) $u_0(x)$ locally satisfies the Hölder condition if $n > 1$.

Some further results on fractional evolution equations can be found in Hilfer.[37]

Remark 6. Mainardi[50] (see also Podlubny[60] or Gorenflo *et al.*[35]) considered the initial-value problem (2.1) and (2.9) for $n = 1$, $\gamma = 0$, $\alpha = 2$, where the fractional-in-time derivative of order $\beta \in (0, 2)$ is interpreted in the Caputo–Djrbashian sense (see Caputo,[16, 17] Mainardi[50] or Anh and Leonenko[8, 7]). He presented the solution of the initial-value problem in the form of (3.20) with the Green function

$$G(t, x) = \frac{1}{2t^{\beta/2}\sqrt{\mu}} M\left(\frac{|x|}{t^{\beta/2}\sqrt{\mu}}; \frac{\beta}{2}\right), \qquad 0 < \beta < 2, \qquad (3.21)$$

where the function

$$M(u; v) = \sum_{j=0}^{\infty} \frac{(-1)^j u^j}{j!\Gamma(-vj+1-v)} = W(-u; -v, 1-v), \qquad u \geqslant 0, \quad 0 < v < 1,$$

and the entire function of order $1/(1+\lambda)$

$$W(z; \lambda, \mu) = \mathscr{J}^{\lambda}_{\mu-1}(z) = H^{1,0}_{0,2}\left(z \,\middle|\, \begin{matrix} - \\ (0, 1)(1-\mu, \lambda) \end{matrix}\right), \qquad z \in \mathbb{C}^1,$$

$\lambda > -1$, $\mu > 0$ is known as Wright's generalized Bessel function (see (C.14) for its definition). The Fourier transform of (3.21) is of the form (2.11) with $\gamma = 0$, $\alpha = 2$, $n = 1$ (see Anh and Leonenko[8] for more details). In particular, $M(u; 1/2) = \exp\{-u^2/4\}$, and $M(u; 1/3)$ can be expressed in terms of Airy function with positive argument.

Remark 7. Saichev and Zaslavski,[67] Zaslavski,[80] Hilfer[37] and Kobelev *et al.*[41] presented some different forms of the Green functions of fractional-in-time and/or in-space one-dimensional diffusion equations. They use a different definition of fractional derivatives and their regularizations.

Remark 8. The Green functions (3.4) or (3.5) of the fractional kinetic equation (2.1) are radial or rotation-invariant, i.e., $G(t, x) = \tilde{G}(t, |x|)$, $t > 0$, $x \in \mathbb{R}^n$. In general, the fundamental solutions of higher-order heat-type equations can be not only signed but also asymmetric (see Fujita,[30] Hochberg and Orsingher[38] or Beghin *et al.*[9] and the references therein). For example, the fundamental solution of the Airy equation or linear Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3}$$

is of the form

$$u(t, x) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right), \qquad t > 0, \quad x \in \mathbb{R}^1, \tag{3.22}$$

where the Airy function of the first kind (see Bleistein and Handelsman[12])

$$Ai(x) = \frac{1}{\sqrt{\pi}} \int^{\infty} cos\left(\alpha x + \frac{\alpha^3}{3}\right) dx, \qquad x \in \mathbb{R}^1$$

is asymmetric and signed. Thus the fundamental solution (3.22) has the following asymptotic behaviour:

$$u(t, x) \sim \frac{x^{-1/4} t^{-1/4}}{2\sqrt{\pi}\sqrt[4]{3}} \exp\left\{-\frac{2}{3\sqrt{3}} x^{3/2} t^{-1/2}\right\}, \qquad x \to +\infty$$

and

$$u(t, x) \sim \frac{|x|^{-1/4}\, t^{-1/4}}{\sqrt{\pi}\sqrt[4]{3}} \cos\left\{\frac{2}{3\sqrt{3}} |x|^{3/2}\, t^{-1/2} - \frac{\pi}{4}\right\}, \qquad x \to -\infty.$$

For any $t > 0$, $u(t, x)$ converges to zero exponentially fast as $x \to +\infty$ and oscillating as $x \to -\infty$. Thus $u(t, x)$ is asymmetric and signed. We note that the fundamental solution (3.1) of the second-order equation is non-negative while that of the fourth-order equation is signed but symmetric. Feller,[29] Fujita,[30] Uchaikin and Zolotarev[77] considered the fractional equations whose fundamental solutions are general densities of stable distributions (not necessarily symmetric). The fundamental solutions $u(t, x)$, $t > 0$, $x \in \mathbb{R}^1$ of the higher-order heat-type equation

$$\frac{\partial u}{\partial t} = (-1)^{m+1} \frac{\partial^{2m} u}{\partial x^{2m}}, \qquad m = 2, 3, \ldots$$

have as their Fourier transforms $\exp\{-\xi^{2m} t\}$ and, for $t = 1$,

$$u(1, x) = \int_{\mathbb{R}^1} \frac{1}{2\pi} \exp\{ix\xi - \xi^{2m}\}\, d\xi = \frac{1}{2\pi m} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma((2j+1)/2m)}{\Gamma(2j+1)} x^{2j}.$$

The function $y(x) = 2\pi u(1, x)$ satisfies the ordinary differential equation

$$\frac{d^{2m-1} y}{dx^{2m-1}} - (-1)^m \frac{1}{2m} xy = 0.$$

which is a special case of Turrittin's equation

$$\frac{d^n y}{dx^n} - x^\nu y = 0, \qquad n = 0, 1, 2, ..., \nu \in \mathbb{C}^1$$

(see Kamimoto[40]).

4. SPECTRAL REPRESENTATION AND SCALING LAWS

In this section we discuss the spectral representation of some classes of random fields which can be interpreted as mean-square solutions of fractional-in-time and in-space kinetic equations (2.1) with random initial condition (2.3). We obtain new Gaussian and non-Gaussian scenarios for renormalized solutions of these equations.

Let $\eta(x)$, $x \in \mathbb{R}^n$, be a real measurable mean-square continuous homogeneous (in the wide sense) random field with $E\eta(x) = 0$ and covariance function

$$B(x) = cov(\eta(0), \eta(x)) = \int_{\mathbb{R}^n} \cos\langle \lambda, x - y \rangle \, F(d\lambda), \tag{4.1}$$

where F is the spectral measure, that is, a bounded non-negative measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, $\mathscr{B}(\mathbb{R}^n)$ being the σ-field of Borel sets of $\mathbb{R}^n$. In view of Karhunen's theorem (see, for example, Gihman and Skorokhod[33]) there exists a complex-valued orthogonally scattered random measure Z such that for every $x \in \mathbb{R}^n$, the random field itself has the spectral representation $(p-$a.s.)

$$\eta(x) = \int_{\mathbb{R}^n} e^{i\langle \lambda, x \rangle} Z(d\lambda), \qquad E\,|Z(A)|^2 = F(A), \qquad A \in \mathscr{B}(\mathbb{R}^n). \tag{4.2}$$

From (2.11), (2.12) and (4.2) we obtain the formal solution of the initial-value problem (2.1) and (2.3) in the form of $L_2(\Omega)$-stochastic integral:

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}^n} \eta(y) \, G(t, x - y) \, dy \\ &= \int_{\mathbb{R}^n} e^{i\langle \lambda, x \rangle} E_\beta(-\mu t^\beta \, |\lambda|^\alpha (1 + |\lambda|^2)^{\gamma/2}) \, Z(d\lambda), \end{aligned} \tag{4.3}$$

where E_β is the Mittag–Leffler function (2.5) and $\beta \in (0, 1]$, $\alpha > 0$, $\gamma \geqslant 0$ are fractional parameters. In addition, by (2.4), (2.8) and (4.3),

$$\begin{aligned}\mathscr{D}_t^\beta u &= -\mu \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} \, |\lambda|^\alpha \, (1+|\lambda|^2)^{\gamma/2} \, E_\beta(-\mu t^\beta \, |\lambda|^\alpha (1+|\lambda|^2)^{\gamma/2}) \, Z(d\lambda) \\ &= -\mu (I-\Delta)^{\gamma/2} \, (-\Delta)^{\alpha/2} \, u,\end{aligned}$$

where the fractional derivatives in space are also interpreted in the mean-square sense in the frequency domain (see Appendix B). Thus we can interpret (4.3) as the mean-square or $L_2(\Omega)$-solution of the initial-value problem (2.1) and (2.3).

The covariance function of the random field (4.3) is of the form

$$\begin{aligned}&cov(u(t, x), u(s, y)) \\ &= \int_{\mathbb{R}^n} e^{i\langle \lambda, x-y\rangle} E_\beta(-\mu t^\beta \, |\lambda|^\alpha \, (1+|\lambda|^2)^{\gamma/2}) \, E_\beta(-\mu s^\beta \, |\lambda|^\alpha \, (1+|\lambda|^2)^{\gamma/2}) \, F(d\lambda).\end{aligned} \tag{4.4}$$

In particular for $\beta = 1$ we obtain the following spectral representation of the mean-square solutions of the fractional-in-space kinetic equation with random data:

$$u(t, x) = \int_{\mathbb{R}^n} \exp \{ i\langle \lambda, x\rangle - \mu t \, |\lambda|^\alpha \, (1+|\lambda|^2)^{\gamma/2} \} \, Z(d\lambda), \tag{4.5}$$

while for $\beta \in (0, 1]$, $\gamma = 0$, $\alpha = 2$, (4.3) reduces to

$$u(t, x) = \int_{\mathbb{R}^n} \exp \{ i\langle \lambda, x\rangle \} \, E_\beta(-\mu t^\beta \, |\lambda|^2) \, Z(d\lambda). \tag{4.6}$$

For $\beta = 1$, $\gamma = 0$, $\alpha = 2$, both spectral representations (4.5) and (4.6) can be written as

$$u(t, x) = \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle - \mu t \, |\lambda|^2} Z(d\lambda). \tag{4.7}$$

Remark 9. Consider the initial-value problem (2.1) and (2.3) with $\beta \in (0, 1]$, $\gamma = 0$, $\alpha = 2$ on the set $(t, x) \in (0, T] \times \mathbb{R}^n$. Suppose that sample paths of the random field $\eta(x)$, $x \in \mathbb{R}^n$ satisfy conditions (1)–(3) of Remark 5. Then there exists a classical solution of the initial-value problem (2.1) and (2.3) (see Remark 5) and (4.3) satisfies (2.1) with probability one. An interesting open problem is to generalize this result to the general equation (2.1).

Remark 10. If $\eta(x)$, $x \in \mathbb{R}^n$, is a homogeneous Gaussian field with spectral representation (4.2), then u(t, x), $t > 0$, $x \in \mathbb{R}^n$, is a homogeneous (in space) Gaussian field with spectral representation (4.3). If the field η is subordinate to a Gaussian field (see Dobrushin[23]), then the field (4.3) can be written as a series of Wiener–Itô multiple integrals with corresponding transfer functions. These transfer functions express the non-Gaussian structure of the field (4.3). In particular, it is possible to calculate the higher-order spectra using the diagram formalism and technique of Terdik.[76] We address these problems in a separate paper.

An open area of investigation is to consider the rescaled solutions of the fractional kinetic equations (2.1) with random initial conditions and/or random potential. For an exposition of the heat equation with random potential in terms of Wiener–Itô integrals, see Nualard and Zakai[56] and Holden *et al.*,[39] for example.

In this paper we shall restrict ourselves to finding the limiting distributions of the rescaled solutions of the initial-value problem (2.1) and (2.3) in the case where the (non-Gaussian) random field $\eta(x) = h(\xi(x))$, $x \in \mathbb{R}^n$, is a local functional of a homogeneous isotropic Gaussian field $\xi(x)$, $x \in \mathbb{R}^n$, such that $E(h^2(\xi(0))) < \infty$. The underlying field $\xi(x)$, $x \in \mathbb{R}^n$, is assumed to satisfy the following conditions:

A. The field $\xi(x)$, $x \in \mathbb{R}^n$, is a real measurable separable mean-square continuous homogeneous isotropic Gaussian random field with $E\xi(x) = 0$ and covariance function of the form

$$R(x) = cov(\xi(0), \xi(x)) = (1 + |x|^2)^{-\kappa/2}, \qquad 0 < \kappa < n, \quad x \in \mathbb{R}^n. \tag{4.8}$$

Observe that in this case

$$\int_{\mathbb{R}^n} B(x)\, dx = \infty$$

and we have a random field with long-range dependence.

Remark 11. Most of the papers devoted to limit theorems for random fields with LRD have used the covariance function of the form $R(x) = L(|x|)/|x|^{\kappa}$, $0 < \kappa < n$, $x \in \mathbb{R}^n$, where L is a slowly varying function for large values of its argument with some additional properties. Nevertheless, for continuous-parameter random fields, it is not easy to find exact examples of non-negative definite continuous functions of the above form. Note that the class of covariance functions of real-valued homogeneous isotropic random fields coincides with the class of characteristic functions of symmetric probability distributions. From the theory of characteristic functions we are currently able to present only example (4.8) and the covariance function

$$R_1(x) = (1 + |x|^{\kappa})^{-1}, \qquad x \in \mathbb{R}^n, \tag{4.9}$$

where $\kappa \in (0, 1)$ for $n = 1$, and $\kappa \in (0, 2)$ for $n \geqslant 2$. The function (4.8) is known as the Fourier transform of the Bessel potential (see Appendix B) or characteristic function of symmetric Bessel distributions (see Oberhettinger,[57] p. 156, or Fang *et al.*,[28] p. 69). The function (4.9) is known as the characteristic function of the Linnik distribution (see Anderson[2] or Ostrovskii[59]). In this paper we consider the covariance function (4.8). In principle, our method is applicable to the covariance function (4.9) as well (see Anh and Loenenko[7] for details).

Under condition A, the covariance function (4.8) has the following spectral representation:

$$R(x) = \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} f_{\kappa}(\lambda)\, d\lambda, \tag{4.10}$$

while the field itself can be represented as

$$\xi(x) = \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} f_{\kappa}^{1/2}(\lambda)\, W(d\lambda), \tag{4.11}$$

where W is the complex-valued white noise random measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ such that

$$E\, |W(\Delta)|^2 = |\Delta|, \qquad \Delta \in \mathscr{B}(\mathbb{R}^n),$$

for any Δ with finite Lebesgue measure $|\Delta|$. The spectral density $f_{\kappa}(\lambda)$, $\lambda \in \mathbb{R}^n$ has the following exact form (see, for example, Donoghue,[25] p. 293):

$$f_{\kappa}(\lambda) = \tilde{f}_{\kappa}(|\lambda|) = c(n, \kappa)\, K_{(n-\kappa)/2}(|\lambda|)\, |\lambda|^{(\kappa - n)/2}, \tag{4.12}$$

where

$$c(n, \kappa) = \left[\pi^{n/2} 2^{(\kappa - n)/2} \Gamma\left(\frac{\kappa}{2}\right)\right]^{-1}$$

and

$$K_\nu(z) = \frac{1}{2}\int_0^\infty s^{\nu-1} e^{-\frac{1}{2}z\,(s+\frac{1}{s})} ds = \frac{1}{2}\, H^{2,\,0}_{0,\,2}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} - \\ (\nu/2,\,1)(-\nu/2,\,1) \end{matrix}\right) \tag{4.13}$$

is the modified Bessel function of the third kind of order ν (see, for example, Watson[78]). We note that

$$K_\nu(z) \sim \Gamma(\nu)\, 2^{\nu-1} z^{-\nu}, \qquad z \downarrow 0, \quad \nu > 0 \tag{4.14}$$

and for a large value of z the following approximation holds:

$$K_\nu(z) = \sqrt{\frac{\pi}{2}}\, e^{-z} z^{-1/2}\left(1 - \frac{4\nu^2 - 1}{8z} + \cdots\right). \tag{4.15}$$

Using (4.12)–(4.15) we obtain the following Tauberian representation (see Donoghue,[25] p. 295):

$$f_\kappa(\lambda) = c_1(n, \kappa)\, |\lambda|^{\kappa-n}\,(1 - \theta(|\lambda|)), \qquad 0 < \kappa < n, \quad \lambda \in \mathbb{R}^n, \tag{4.16}$$

where $\theta(|\lambda|) \to 0$ as $|\lambda| \to 0$, and

$$c_1(n, \kappa) = \Gamma\left(\frac{n-\kappa}{2}\right) \Big/ \left[2^\kappa \pi^{n/2}\, \Gamma\left(\frac{\kappa}{2}\right)\right] \tag{4.17}$$

is a Tauberian constant (see Leonenko,[45] p. 67).

Remark 12. The correlation function (4.9) has the spectral representation (4.10) with spectral density

$$f_{1\kappa}(\lambda) = \frac{\sin(\pi\kappa/2)}{2^{(n-2)/2}\pi^{(n+2)/2}}\, |\lambda|^{(2-n)/2} \int_0^\infty K_{(n-2)/2}(|\lambda|\, u)\, \frac{u^{(n/2)+\kappa} du}{|1 + u^\kappa e^{i\pi\kappa/2}|^2}, \qquad \lambda \in \mathbb{R}^n$$

for which we can derive the Tauberian representation (4.16), but its asymptotic behaviour at the origin depends on the arithmetic nature of the parameter $\kappa \in (0, 2)$ (see Ostrovskii[59]).

From (4.16) we observe that $f_\kappa(\lambda) \uparrow \infty$ as $|\lambda| \to 0$; thus we have a field with singular spectrum.

B. The function h: $\mathbb{R}^1 \to \mathbb{R}^1$ is such that

$$\int_{\mathbb{R}^1} h^2(u)\, \varphi(u)\, du < \infty,$$

where

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \qquad u \in \mathbb{R}^1$$

is the standard Gaussian density.

The (non-linear) function h of condition B may be expanded in the series

$$h(u) = \sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(u), \qquad C_k = \int_{\mathbb{R}^1} h(u)\, \varphi(u)\, H_k(u)\, du, \tag{4.18}$$

of orthogonal Chebyshev–Hermite polynomials

$$H_k(u) = (-1)^k\, [\varphi(u)]^{-1} \frac{d^k}{du^k} \varphi(u), \qquad k = 0, 1, 2, \ldots$$

C. There exists an integer $m \geqslant 1$ such that

$$C_0 = \cdots = C_{m-1} = 0, \qquad C_m \neq 0.$$

The integer $m \geqslant 1$ will be called the Hermitian rank of the function h (see, for example, Taqqu[75]).

D. Suppose that the Green function $G(t, \cdot) \in L_1(\mathbb{R}^n)$.

Remark 13. For a discussion of condition D in terms of fractional parameters, see Section 3. For example, condition D holds if $\alpha + \gamma > n$, $\beta \in (0, 1]$.

Our main result is the following

Theorem 3. Let $u(t, x)$, $0 < t \leqslant T$, $x \in \mathbb{R}^n$, be a random field of the form (4.3), where $\alpha > 0$, $\beta \in (0, 1]$, $\gamma \geqslant 0$ are fractional parameters of the fractional kinetic equation (2.1) with the initial condition field $\eta(x) = h(\xi(x))$, $x \in \mathbb{R}^n$, where the non-random function h and random field $\xi(x)$, $x \in \mathbb{R}^n$, satisfy conditions A, B and C with

$$\kappa < \min(2\alpha, n)/m,$$

$m \geqslant 1$ being the Hermitian rank of the function h. Suppose that condition D holds. Then the finite-dimensional distributions of the random field

$$U_\varepsilon(t, x) = \frac{1}{\varepsilon^{m\kappa\beta/(2\alpha)}}\, u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right), \qquad 0 < t \leqslant T, \quad x \in \mathbb{R}^n, \tag{4.19}$$

converge weakly as $\varepsilon \to 0$ to the finite-dimensional distributions of the random field

$$U_m(t, x) = \frac{C_m}{m!} c_1^{m/2}(n, \kappa) \int'_{\mathbb{R}^{nm}} \frac{e^{i\langle x, \lambda_1 + \cdots + \lambda_m \rangle}}{(|\lambda_1| \cdots |\lambda_m|)^{(n-\kappa)/2}} E_\beta(-\mu t^\beta |\lambda_1 + \cdots + \lambda_m|^\alpha)$$
$$\times W(d\lambda_1)...W(d\lambda_m), \qquad 0 < t \leqslant T, \quad x \in \mathbb{R}^n, \tag{4.20}$$

where E_β is the Mittag–Leffler function (2.5), W is the complex Gaussian white noise random measure defined by (4.11) and $c_1(n, \kappa)$ is a constant defined in (4.17).

We shall give the proof in Section 5. Here $\int' ...$ is multiple Wiener–Itô integral with respect to a Gaussian white noise measure. For definition and properties of these integrals, see Taqqu,[75] Major,[53] for example. We should note that the diagonal hyperplanes $\lambda_i = \pm\lambda_j$, $i, j = 1, ..., m$, $i \neq j$, are excluded from the domain of integration. The random field (4.20) is homogeneous in $x \in \mathbb{R}^n$, that is,

$$EU_m(t, x)\, U_m(s, y) = \frac{C_m^2}{m!} c^m(n, \kappa) \int_{\mathbb{R}^{nm}} \frac{e^{i\langle x-y, \lambda_1 + \cdots + \lambda_m \rangle}}{(|\lambda_1|...|\lambda_m|)^{n-\kappa}} E_\beta(-\mu t^\beta\, |\lambda_1 + \cdots + \lambda_m|^\alpha)$$
$$\times E_\beta(-\mu s^\beta\, |\lambda_1 + \cdots + \lambda_m|^\alpha)\, d\lambda_1...d\lambda_m. \tag{4.21}$$

It is easy to see that $EU^2(t, x) < \infty$ if $\kappa < \min(2\alpha, n)/m$ (see (3.3)).

Note that Theorem 3 reduces to the result of Leonenko and Woyczynski[47] for $n = 1$, $\beta = 1$, $\gamma = 0$, $\alpha = 2$, to the result of Anh and Leonenko[6] for $n \geqslant 1$, $\beta = 1$, $\gamma = 0$, $\alpha = 2$, to the result of Anh and Leonenko[8] for $n = 1$, $\beta \in (0, 1]$, $\gamma = 0$, $\alpha = 2$. The special cases $n \geqslant 1$, $\beta \in (0, 1]$, $\gamma = 0$, $\alpha = 2$, and $n \geqslant 1$, $\beta = 1$, $\gamma \geqslant 0$, $\alpha > 0$ of Theorem 3 were considered in Anh and Leonenko[7].

We observe from (4.20) that $U_1(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, is a homogeneous (in x) Gaussian random field with covariance function (4.21) for $m = 1$ and spectral density

$$g(\lambda) = c_2 E_\beta^2(-\mu t^\beta\, |\lambda|^\alpha)\, |\lambda|^{\kappa - n}, \qquad \lambda \in \mathbb{R}^n, \tag{4.22}$$

where

$$c_2 = C_1^2 c_1(n, \kappa).$$

The spectral density (4.22) behaves as

$$c_2 |\lambda|^{\kappa - n}, \qquad \kappa \in (0, \min(2\alpha, n)), \tag{4.23}$$

as $|\lambda| \to 0$. Hence the Gaussian random field $U_1(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, which can be considered as an approximation to the solution of the fractional kinetic equation with random singular data, displays LRD.

Applying (2.7) to (4.22) yields

$$g(\lambda) = \frac{c_2}{\mu t^\beta} \frac{1}{|\lambda|^{n+2\alpha-\kappa}} + O\left(\frac{1}{|\lambda|^{n+2\alpha-\kappa+1}}\right) \tag{4.24}$$

as $|\lambda| \to \infty$. The component $1/|\lambda|^{n+2\alpha-\kappa}$ indicates the second-order intermittency (see Anh *et al.*[(5)]). The component $t^{-\beta}$ indicates that the relaxation function is non-exponential.

The random fields $U_m(t, x)$, $m \geqslant 2$, have a non-Gaussian structure. In principle, it is possible to calculate the higher-order spectral densities of these fields based on the diagram formalism (see Dobrushin[(23)]) and the special technique of Terdik ([(76)]). These higher-order spectral densities of non-Gaussian fields $U_m(t, x)$, $m \geqslant 2$, also have singularities at frequency zero (and on the diagonals $\lambda_i = -\lambda_j$, $i, j = 1, \ldots, m$) and intermittency-type behaviour at infinity. We address these problems in a separate paper.

The above discussion shows that random fields U_m, defined in (4.20), which can be considered via Theorem 3 as approximations to the solutions of the fractional kinetic equation (2.1) with singular data, can be used as models of physical phenomena with important features such as non-Gaussian marginal distributions, LRD, intermittency and non-exponential relaxation simultaneously. Moreover the explicit form of the spectral densities such as (4.22) leads to suitable methods for statistical estimation of the parameters of these random fields in the frequency domain (see Leonenko[(45)] or Leonenko and Woyczynski,[(48)] among others) and for their simulation.

5. PROOF OF THE MAIN RESULT

Under the conditions of Theorem 3, we have the following Hermite expansion in $L_2(\Omega)$:

$$u(t, x) = \int_{\mathbb{R}^n} G(t, x-y)\, h(\xi(y))\, dy = \sum_{k=m}^{\infty} \frac{C_k}{k!} v_k(t, x), \tag{5.1}$$

where by Itô's formula

$$v_k(t, x) = \int_{\mathbb{R}^n} G(t, x-y)\, H_k(\xi(y))\, dy$$

$$= \int'_{\mathbb{R}^{nm}} e^{i\langle x, \lambda_1 + \cdots + \lambda_m \rangle} E_\beta(-\mu t^\beta\, |\lambda_1 + \cdots + \lambda_m|^\alpha\, (1 + |\lambda_1 + \cdots + \lambda_m|^2)^{\gamma/2})$$

$$\times \prod_{j=1}^{m} f_\kappa^{1/2}(\lambda_j)\, W(d\lambda_1)...W(d\lambda_m), \tag{5.2}$$

where $\int'$ means Wiener–Itô integral.

Therefore

$$U_\varepsilon(t, x) = \frac{1}{\varepsilon^{m\kappa\beta/2\alpha}}\, u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right) = \zeta_{m,\varepsilon}(t, x) + R_\varepsilon, \tag{5.3}$$

where by the scaling property of Gaussian white noise ($W(ad\lambda) \overset{d}{=} a^{n/2} W(d\lambda)$, where $\overset{d}{=}$ stands for equality of distributions) we have

$$\zeta_{m,\varepsilon}(t, x) = \frac{(C_m/m!)}{\varepsilon^{m\kappa\beta/(2\alpha)}}\, v_m\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right)$$

$$= \frac{(C_m/m!)}{\varepsilon^{m\kappa\beta/(2\alpha)}} \int'_{\mathbb{R}^{nm}} e^{i\langle x\varepsilon^{-\beta/\alpha}, \lambda_1 + \cdots + \lambda_m \rangle} E_\beta(-\mu(t/\varepsilon)^\beta\, |\lambda_1 + \cdots + \lambda_m|^\alpha$$

$$\times (1 + |\lambda_1 + \cdots + \lambda_m|^2)^{\gamma/2})$$

$$\times \prod_{j=1}^{m} f_\kappa^{1/2}(\lambda)\, W(d\lambda_1)...W(d\lambda_m)$$

$$\overset{d}{=} \frac{(C_m/m!)}{\varepsilon^{m\kappa\beta/(2\alpha)}} \int'_{\mathbb{R}^{nm}} e^{i\langle x, \lambda_1 + \cdots + \lambda_m \rangle} E_\beta(-\mu(t/\varepsilon)^\beta\, |\lambda_1 + \cdots + \lambda_m|^\alpha$$

$$\times \varepsilon^\beta (1 + \varepsilon^{2\beta/\alpha}\, |\lambda_1 + \cdots + \lambda_m|^2)^{\gamma/2})$$

$$\times \prod_{j=1}^{m} f_\kappa^{1/2}(\lambda_j \varepsilon^{\beta/\alpha})\, \varepsilon^{mn\beta/(2\alpha)} W(d\lambda_1)...W(d\lambda_m)$$

$$= \frac{C_m}{m!} \int_{\mathbb{R}^{nm}}' e^{i\langle x, \lambda_1 + \cdots + \lambda_m \rangle} E_\beta(-\mu t^\beta \, |\lambda_1 + \cdots + \lambda_m|^\alpha$$
$$\times (1 + \varepsilon^{2\beta/\alpha} \, |\lambda_1 + \cdots + \lambda_m|^2)^{\gamma/2}) \, c_1^{m/2}(n, \kappa)$$
$$\times \left\{ \prod_{j=1}^m |\lambda_j|^{(\kappa - n)/2} \right\} \left\{ \prod_{j=1}^m (1 - \theta(|\lambda_j| \, \varepsilon^{\beta/\alpha})) \right\} W(d\lambda_1)...W(d\lambda_m), \tag{5.4}$$

where $\theta(|\lambda|)$ is defined in (4.16).

We shall prove that

$$\Delta_\varepsilon = E \, |\zeta_{m,\varepsilon}(t, x) - U_m(t, \varepsilon)|^2 \to 0, \qquad \kappa m < \min(2\alpha, n), \tag{5.5}$$

and

$$var R_\varepsilon = \frac{1}{\varepsilon^{m\kappa\beta/(2\alpha)}} \, var \sum_{k=m}^\infty \frac{C_k}{k!} \, v_k\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right) \to 0 \tag{5.6}$$

as $\varepsilon \to 0$. Then, by Slutsky's argument, $U_{m,\varepsilon}(t, x)$ converges in distribution to $U_m(t, x)$ and the statement of Theorem 3 can be obtained by the Cramer–Wold argument.

From (5.4) and (4.20) we have

$$\Delta_\varepsilon \leqslant \frac{C_m^2}{m!} \, c_1^m(n, \kappa) \int_{\mathbb{R}^{nm}} \frac{|e^{i\langle x, \lambda_1 + \cdots + \lambda_m \rangle} E_\beta(-\mu t^\beta \, |\lambda_1 + \cdots + \lambda_m|^\alpha)|^2}{(|\lambda_1 + \cdots + \lambda_m|)^{n-\kappa}}$$
$$\times Q_\varepsilon(\lambda_1, ..., .\lambda_m) \, d\lambda_1...d\lambda_m, \tag{5.7}$$

where

$$Q_\varepsilon(\lambda_1, ..., .\lambda_m) = \Pi_{j=1}^m (1 - \theta_j(|\lambda_j| \, \varepsilon^{\beta/\alpha})) \, E_\beta^{-2}(-\mu t^\beta |\lambda_1 + \cdots + \lambda_m|^\alpha)$$
$$\times E_\beta^2(-\mu t^\beta \, |\lambda_1 + \cdots + \lambda_m|^\alpha \, (1 + \varepsilon^{\beta/\alpha} \, |\lambda_1 + \cdots + \lambda_m|^2)^{\gamma/2}) - 1.$$

Using the property of complete monotonicity of Mittag–Leffler function E_β with $\beta < 1$ and (4.12), (4.14), (4.15) and (4.16) it follows that $Q_\varepsilon(\lambda_1, ..., .\lambda_m)$ is a bounded function and $\lim_{\varepsilon \to 0} Q_\varepsilon(\lambda_1, ..., .\lambda_m) = 0$. Then by the dominated convergence theorem, we get from (5.7) $\lim_{\varepsilon \to 0} \Delta_\varepsilon = 0$. Note that from (2.7) it follows that

$$\int_{\mathbb{R}^{nm}} \frac{|e^{i\langle x, \lambda_1 + \cdots + \lambda_m \rangle} E_\beta(-\mu t^\beta \, |\lambda_1 + \cdots + \lambda_m|^2)|^2}{(|\lambda_1 + \cdots + \lambda_m|)^{n-\kappa}} \, d\lambda_1...d\lambda_m < \infty,$$

for $\kappa < \min(2\alpha, n)/m$. Thus, (5.5) holds.

Let us now prove (5.6). It is well known that

$$EH_k(\xi(x))\, H_m(\xi(y)) = m!\delta_k^m [R(x)]^m, \tag{5.8}$$

where R is defined by (4.8) and δ_k^m is the Kronecker symbol. Using (5.8) we get

$$var \sum_{k=m}^{\infty} \frac{C_k}{k!} v_k\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right) \leqslant A_{m+1}(t, x) \sum_{k \geqslant m+1} \frac{C_k^2}{k!}, \tag{5.9}$$

where

$$\begin{aligned} A_{m+1,\varepsilon}(t, x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_1\right)\right| \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_2\right)\right| R^{m+1}(y_1 - y_2)\, dy_1\, dy_2 \\ &= \int_{\upsilon(\varepsilon^{-2/\alpha})}\int_{\upsilon(\varepsilon^{-2/\alpha})} \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_1\right)\right| \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_2\right)\right| \\ &\quad \times R^{m+1}(y_1 - y_2)\, dy_1 dy_2 + W_\varepsilon(t, x) \\ &= A'_{m+1,\varepsilon} + W_\varepsilon, \end{aligned} \tag{5.10}$$

and

$$\upsilon(r) = \{x \in \mathbb{R}^n : |x|_0 < r\}, \qquad |x|_0 = \max\{|x_1|, ..., |x_n|\}.$$

Note that

$$G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}}\right) = (2\pi)^{-n}\, \varepsilon^{n\beta/\alpha} \int_{\mathbb{R}^n} e^{i\langle \lambda, x - y\varepsilon^{\beta/\alpha}\rangle} E_\beta(-\mu t^\beta\, |\lambda|^\alpha\, (1 + \varepsilon^{2\beta/\alpha}\, |\lambda|^2)^{\gamma/2})\, d\lambda. \tag{5.11}$$

For any $b > 0$ there exists $U > 0$ such that $R(y_1 - y_2) < b$ if $|y_1 - y_2|_0 > U$. Now, let

$$\begin{aligned} \Delta_1 &= \{y_1 \in \upsilon(\varepsilon^{-2/\alpha}),\, y_2 \in \upsilon(\varepsilon^{-2/\alpha}) : |y_1 - y_2| < U\}, \\ \Delta_2 &= \upsilon(\varepsilon^{-2/\alpha}) \times \upsilon(\varepsilon^{-2/\alpha}) \setminus \Delta_1, \\ \Delta_3 &= \{z_i = x - y_i \varepsilon^{\beta/\alpha},\, y_i \in \upsilon(\varepsilon^{-2/\alpha}),\, i = 1.2 : |z_1 - z_2|_0 > U\varepsilon^{\beta/\alpha}\}. \end{aligned}$$

Then from (5.10) and (5.11) we get

$$\begin{aligned} A'_{m+1,\varepsilon} &= \int_{\Delta_1}\int \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_1\right)\right| \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_2\right)\right| dy_1 dy_2 \\ &\quad + b\int_{\Delta_2}\int \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_1\right)\right| \left|G\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/\alpha}} - y_2\right)\right| R^m(y_1 - y_2)\, dy_1 dy_2 \\ &\leqslant \varepsilon^{2n\beta/\alpha}\psi_1(\varepsilon) + b\int_{\Delta_3}\int \left|(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle\lambda, x - z_1\rangle}\right. \\ &\quad \left. \times E_\beta(-\mu t^\beta\, |\lambda|^\alpha\, (1 + \varepsilon^{2\beta/\alpha}\, |\lambda|^2)^{\gamma/2})\, d\lambda\right| \\ &\quad \times \left|(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle\lambda, x - z_2\rangle} E_\beta(-\mu t^\beta\, |\lambda'|^\alpha\, (1 + \varepsilon^{2\beta/\alpha}\, |\lambda'|^2)^{\gamma/2})\, d\lambda'\right| \\ &\quad \times \frac{\varepsilon^{m\beta/\alpha}}{|\varepsilon^{2\beta/\alpha}\, |z_1 - z_2|^2|^{m\kappa/2}}\, dz_1 dz_2 \\ &\leqslant \varepsilon^{2n\beta/\alpha}\psi_1(\varepsilon) + b\varepsilon^{m\kappa\beta/\alpha}\psi_2(\varepsilon), \end{aligned} \tag{5.12}$$

where, by the dominated convergence theorem, $\lim_{\varepsilon \to 0} \psi_i(\varepsilon) = K_i$, K_i, $i = 1, 2$, being positive constants. From (5.11) we obtain

$$A'_{m+1}/\varepsilon^{m\kappa\beta/\alpha} \leqslant \psi_1(\varepsilon)\, \varepsilon^{(\beta/\alpha)(2n - m\kappa)} + b\psi_2(\varepsilon) \tag{5.13}$$

The right-hand side of (5.13) tends to zero as $\varepsilon \to 0$ since $\kappa m < 2n$ and $b > 0$ can be chosen arbitrary small.

Making again some change of variables we can prove that

$$W_\varepsilon/\varepsilon^{m\kappa\beta/\alpha} \to 0, \qquad \varepsilon \to 0. \tag{5.14}$$

From (5.9)–(5.14) we obtain (5.6). The proof of Theorem 3 is then completed.

A. CAPUTO–DJRBASHIAN'S REGULARIZED FRACTIONAL DERIVATIVE

This appendix is based on Caputo,[16] Djrbashian and Nersesian[22] and Djrbashain (ref. 21, Chapter 10), Podlubny,[60] Butzer and Westphal.[15] It should be mentioned that Caputo and Djrbashian independently developed the concept of regularized fractional derivative without naming it explicitly. Later, the concept was re-discovered by several authors.

Let $f(t) \in L_1(0,T)$ be an arbitrary function. The Riemann–Liouville fractional integral of order $\beta > 0$ is defined as

$$\mathscr{R}^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau, \qquad 0 < \tau < T. \tag{A.1}$$

For any $\beta > 0$ the functions $\mathscr{R}^{-\beta} f(t) \in L_1(0,T)$ and defined almost everywhere. Moreover

$$\lim_{\beta \to 0} \mathscr{R}^{-\beta} f(t) = f(y), \qquad t \in E_f,$$

where E_f is the set of those points $t \in (0,T)$ for which the functions $f(t)$ and $|f(t)|$ are both the derivatives of their primitives (the measure of $(0,T) \setminus E_f$ is zero). For example

$$\mathscr{R}^{-\beta}\{t^a/\Gamma(1+a)\} = t^{a+\beta}/\Gamma(1+a+\beta), \qquad a > -1. \tag{A.2}$$

Let $f(t) \in L_1(0,T)$ and let $\beta_1, \beta_2 \geqslant 0$. Then

$$\begin{aligned} \mathscr{R}^{-\beta_2}(\mathscr{R}^{-\beta_1} f(t)) &= \mathscr{R}^{-\beta_2}(\mathscr{R}^{-\beta_1} f(t)) = \mathscr{R}^{-(\beta_1+\beta_2)} f(t), \\ \mathscr{R}^{-0} f(t) &= f(t). \end{aligned} \tag{A.3}$$

Assume $\beta > 0$ to be a given number and the integer $p \geqslant 1$ to be defined by the inequalities $p-1 < \beta \leqslant p$. For $f(t) \in L_1(0,T)$ we introduce the function

$$\mathscr{R}^{\beta} f(t) = \frac{d^p}{dt^p} \{\mathscr{R}^{-(p-\beta)} f(t)\}$$

which is called the Riemann–Liouville fractional derivative of order $\beta > 0$. In particular, for $p = 1$

$$\mathscr{R}^{\beta} f(t) = \frac{d}{dt} \{\mathscr{R}^{-(1-\beta)} f(t)\}, \qquad 0 < \beta \leqslant 1.$$

For $\beta = 0$ we formally set

$$\mathscr{R}^{0} f(t) = \frac{d}{dt} \{\mathscr{R}^{-1} f(t)\} = f(t).$$

In particular, using (A.2) gives

$$\mathscr{R}^{\beta}\{t^{a}/\Gamma(1+a)\} = t^{a-\beta}/\Gamma(1+a-\beta), \qquad a > -1.$$

Let $AC(0, T)$ be the space of absolutely continuous functions on $(0, T)$. If $f(t) \in L_1(0, T)$, then almost everywhere in $t \in (0, T)$

$$\mathscr{R}^{\beta_1}(\mathscr{R}^{-\beta_2} f(t)) = \mathscr{R}^{-(\beta_2-\beta_1)} f(t), \qquad \beta_2 \geqslant \beta_1 \geqslant 0$$

and

$$\mathscr{R}^{\beta_1}(\mathscr{R}^{-\beta_2} f(t)) = \mathscr{R}^{\beta_1-\beta_2} f(t), \qquad \beta_1 > \beta_2 \geqslant 0$$

if the derivative $\mathscr{R}^{\beta_1-\beta_2} f(t)$ exists almost everywhere in $(0, T)$.

If $f(t) \in L_1(0, T)$ and, in addition,

$$\mathscr{R}^{-(p-\beta_2)} f(t) \in AC^{p}(0, T), \qquad p-1 < \beta_2 \leqslant p, \quad p \geqslant 1,$$

then the equality

$$\mathscr{R}^{-\beta_1}\mathscr{R}^{\beta_2} f(t) = \mathscr{R}^{\beta_2-\beta_1} f(t) - \sum_{k=1}^{p} \{\mathscr{R}^{\beta_2-k} f(\tau)\}|_{\tau=0} \frac{t^{\beta_1-k}}{\Gamma(1+\beta_1-k)} \tag{A.4}$$

is true almost everywhere in $(0, T)$ for any $\beta_1 > 0$. In particular, from (A.4) with $\beta_1 = 1-\beta$, $\beta_2 = 1$, we obtain the Caputo–Djrbashian regularized fractional derivative

$$\begin{aligned} \mathscr{D}^{\beta} f(t) &= \mathscr{R}^{-(1-\beta)} \frac{d}{dt} f(t) \\ &= \frac{1}{\Gamma(1-\beta)} \left[\frac{d}{dt} \int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\beta}} - \frac{f(0)}{t^{\beta}} \right] \\ &= (\mathscr{R}^{\beta} f)(t) - \frac{f(0)}{t^{\beta}\Gamma(1-\beta)}. \end{aligned}$$

B. THE BESSEL AND RIESZ POTENTIALS

This appendix is based on Donoghue,[25] Stein,[73] and Anh *et al.*[5] The integral operator

$$\mathscr{I}_{\gamma} = (I-\Delta)^{-\gamma/2}$$

for $\gamma \in \mathbb{R}_+$ is called the Bessel potential of order γ, whose kernel I_γ is given by

$$I_\gamma(x) = \frac{1}{(4\pi)^{\gamma/2}} \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-\pi |x|^2/s} e^{-s/4\pi} s^{(-n+\gamma)/2} \frac{ds}{s}.$$

Here, Δ is the Laplacian. The following proposition gives some fundamental properties of Bessel potentials.

Proposition 1. For each $\gamma \in \mathbb{R}_+$, $I_\gamma(x) \in L_1(\mathbb{R}^n)$, and its Fourier transform is

$$\hat{I}_\gamma(\lambda) = (2\pi)^{-n/2}(1+|\lambda|^2)^{-\gamma/2}, \qquad \lambda \in \mathbb{R}^n.$$

For $f \in L_p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$,

$$\mathscr{I}_\gamma(f) = I_\gamma * f$$

(the convolution of I_γ and f), and

$$I_\gamma * I_\beta = I_{(\gamma+\beta)}.$$

Therefore,

$$\mathscr{I}_\gamma \cdot \mathscr{I}_\beta = \mathscr{I}_{(\gamma+\beta)}, \qquad \gamma \geqslant 0, \quad \beta \geqslant 0.$$

On the other hand, the inverse of operator $\mathscr{I}_\gamma$ is the operator $\mathscr{I}_{-\gamma} = (I-\Delta)^{\gamma/2}$, for $\gamma \geqslant 0$.

Proof. See Stein,[73] pp. 130–135. ∎

The Riesz potential is defined by $\mathscr{J}_\alpha = (-\Delta)^{-\alpha/2}$, $0 < \alpha < n$. Then, for $f \in \mathscr{S}(\mathbb{R}^n)$,

$$\mathscr{J}_\alpha(f)(x) = \frac{1}{g(\alpha)} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy$$
$$= (J_\alpha * f)(x),$$

where

$$g(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(n/2-\alpha/2)},$$

and

$$J_\alpha(t) = \frac{|t|^{\alpha-n}}{g(\alpha)}$$

is the Riesz kernel, whose Fourier transform is

$$\hat{J}_\alpha(\lambda) = (2\pi)^{-n/2} |\lambda|^{-\alpha}, \qquad \lambda \in \mathbb{R}^n.$$

C. FOX'S *H*-FUNCTIONS

We reproduce the definition and the basic properties of Fox's H-functions (see Braaksma,[13] Srivastava *et al.*[72] or Prudnikov *et al.*[61]).

Fox's H-functions are defined for $z \in \mathbb{C}$, $z \neq 0$ by the Mellin–Barnes-type integral

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}(a_1, \alpha_1)\ldots(a_p, \alpha_p)\\(b_1, \beta_1)\ldots(b_q, \beta_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_L h(s) z^{-s}\, ds, \qquad \text{(C.1)}$$

where $h(s)$ is given by

$$h(s) = A(s)\, B(s)/(C(s)\, D(s))$$

with

$$A(s) = \Pi^m_{j=1}\Gamma(b_j + \beta_j s), \qquad B(s) = \Pi^n_{j=1}\Gamma(1 - a_j - \alpha_j s),$$

$$C(s) = \Pi^q_{j=m+1}\Gamma(1 - b_j - \beta_j s), \qquad D(s) = \Pi^p_{j=n+1}\Gamma(a_j + \alpha_j s).$$

The integers m, n, p and q satisfy $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q$, and empty products are interpreted as unity. The parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are positive real numbers, whereas $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ are complex numbers.

In (C.1) $z^{-s} = \exp\{-s \log|z| - i \arg z\}$ and $\arg z$ is not neccessarily the principal value. The parameters are restricted by the condition $\mathscr{P}(A) \cap \mathscr{P}(B) = \varnothing$, where

$$\mathscr{P}(A) = \{\text{poles of } \Gamma(1 - a_i + \alpha_i s)\} = \left\{\frac{1 - a_i + k}{\alpha_i} \in \mathbb{C};\ i = 1, \ldots, n,\ k \in \mathbb{N}_0\right\},$$

$$\mathscr{P}(B) = \{\text{poles of } \Gamma(b_i + \beta_i s)\} = \left\{\frac{-b_i - k}{\beta_i} \in \mathbb{C};\ i = 1, \ldots, m,\ k \in \mathbb{N}_0\right\},$$

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

The integral (C.1) converges if one of the following conditions holds (Prudnikov *et al.*,[61] Braaksma,[13] Hilfer[37]):

$$L = L(c - i\infty, c + i\infty; \mathscr{P}(A), \mathscr{P}(B)),$$
$$|\arg z| < \omega\pi/2, \quad \omega > 0; \tag{C.2a}$$

$$L = L(c - i\infty, c + i\infty; \mathscr{P}(A), \mathscr{P}(B)),$$
$$|\arg z| = \omega\pi/2, \quad \omega \geqslant 0, \quad cM < -\mathrm{Re}\,\gamma; \tag{C.2b}$$

$$L = L(-\infty + i\mu_1, -\infty + i\mu_2; \mathscr{P}(A), \mathscr{P}(B)),$$
$$M > 0, \quad 0 < |z| < \infty; \tag{C.2c}$$

$$L = L(-\infty + i\mu_1, -\infty + i\mu_2; \mathscr{P}(A), \mathscr{P}(B)),$$
$$M = 0, \quad 0 < |z| < R; \tag{C.2d}$$

$$L = L(-\infty + i\mu_1, -\infty + i\mu_2; \mathscr{P}(A), \mathscr{P}(B)),$$
$$M = 0, \quad |z| = R, \quad \omega \geqslant 0, \quad \mathrm{Re}\,\gamma < 0; \tag{C.2e}$$

$$L = L(\infty + i\mu_1, \infty + i\mu_2; \mathscr{P}(A), \mathscr{P}(B)),$$
$$M < 0, \quad 0 < |z| < \infty; \tag{C.2f}$$

$$L = L(\infty + i\gamma_1, \infty + i\gamma_2; \mathscr{P}(A), \mathscr{P}(B)),$$
$$M = 0, \quad |z| > R; \tag{C.2g}$$

$$L = L(\infty + i\gamma_1, \infty + i\gamma_2; \mathscr{P}(A), \mathscr{P}(B)),$$
$$M = 0, \quad |z| = R, \quad \omega \geqslant 0, \ \mathrm{Re}\,\gamma < 0, \tag{C.2h}$$

where $\mu_1 < \mu_2$. Here $L(z_1, z_2; G_1, G_2)$ denotes a contour in the complex plane starting at $z_{1,}$ ending at z_2 and separating the points in G_1 from those in G_2, and the following expressions are employed:

$$\omega = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j,$$

$$M = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0,$$

$$R = \prod_{j=1}^{p} \alpha_j^{-\alpha_j} \prod_{i=1}^{q} \beta_i^{\beta_i},$$

$$\gamma = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + (p-q)/2 + 1.$$

The H-functions are analytic for $z \neq 0$ and multivalued (single-valued on the Riemann surface of $\log z$). The H-functions may be represented as the series (Braaksma,[13] Hilfer[37])

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}(a_1,\alpha_1)\dots(a_p,\alpha_p)\\(b_1,\beta_1)\dots(b_q,\beta_q)\end{matrix}\right)=\sum_{i=1}^{m}\sum_{k=0}^{\infty} c_{ik}\frac{(-1)^k}{k!\beta_i}\, z^{(b_i+k)/\beta_i}, \tag{C.3}$$

where

$$c_{ik}=\frac{\prod_{j=1,j\neq i}^{m}\Gamma(b_j-(b_i+k)\,\beta_j/\beta_i)\prod_{j=1}^{n}\Gamma(1-a_j+(b_i+k)\,\alpha_j/\beta_i)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+(b_i+k)\,\beta_j/\beta_i)\prod_{j=n+1}^{p}\Gamma(a_j-(b_i+k)\,\alpha_j/b_i)}$$

whenever $M \geqslant 0$, L is as in (C.2a), (C.2b) or (C.2c), (C.2d), (C.2e) and the poles in $\mathscr{P}(A)$ are simple.

Similarly,

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}(a_1,\alpha_1)\dots(a_p,\alpha_p)\\(b_1,\beta_1)\dots(b_q,\beta_q)\end{matrix}\right)=\sum_{i=1}^{m}\sum_{k=0}^{\infty} c_{ik}\frac{(-1)^k}{k!\alpha_i}\, z^{-(1+\alpha_i+k)/\alpha_i}, \tag{C.4}$$

where

$$c_{ik}=\frac{\prod_{j=1,j\neq i}^{n}\Gamma(1-a_j-(1-a_i+k)\,\alpha_j/\alpha_i)\prod_{j=1}^{m}\Gamma(b_j+(1-a_i+k)\,\beta_j/\alpha_i)}{\prod_{j=n+1}^{p}\Gamma(a_j+(1-a_i+k)\,\alpha_j/\alpha_i)\prod_{j=m+1}^{q}\Gamma(1-b_j-(1-a_i+k)\,\beta_j/\alpha_i)}$$

whenever $M \leqslant 0$, L is as given in (C.2a), (C.2b) or (C.2f), (C.2g), (C.2h) and the poles in $\mathscr{P}(A)$ are simple. In particular, if $M>0$, we obtain from (C.3) that

$$H^{1,n}_{p,q}(z)=\frac{1}{\beta_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{B(s_k)}{C(s_k)\,D(s_k)}\, z^{-s_k}, \tag{C.5}$$

where $s_k=-(k+b_1)/\beta_1$, $k \in \mathbb{N}_0$. To get this representation, one transforms the contour L into the left loop and uses the residue theorem with the "left" poles. In the case $M<0$ one can transform the contour L into the right loop and uses the residue theorem with the "right" poles to get

$$H^{m,1}_{p,q}(z)=\frac{1}{\alpha_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{A(s_k)}{C(s_k)\,D(s_k)}\, z^{-s_k}, \tag{C.6}$$

where $s_k=(k+1-a_1)/\alpha_1$, $k \in \mathbb{N}_0$.

For $n=0$, there are cases where H-functions become exponentially small in certain sectors when $|z|$ becomes large (see Braaksma,[13] Eqs. (2.16), (2.36), (2.43)). For $m=q$, we have

$$H^{q,0}_{p,q}(z) \sim F z^{\gamma/M} \exp\{-z^{1/M} M R^{-1/M}\} \tag{C.7}$$

for large $|z|$, uniformly on every closed sector with vertex at the origin contained in $|\arg z| < \omega\pi/2$, where

$$\gamma = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + (p-q+1)/2,$$

$$F = (2\pi)^{(p-q+1)/2} R^{-\gamma/M} M^{-1/2} \prod_{j=1}^{p} \alpha_j^{(1/2)-a_j} \prod_{j=1}^{q} \beta_j^{b_j-(1/2)}.$$

Symmetries in the parameters of the H-function are detected by regarding the definition (C.1). For example, the H-function is symmetric in the set of pairs $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$, in $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$, in $(b_1, \beta_1), \ldots, (b_m, \beta_m)$ and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$.

We give the following reduction formula:

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1)\ldots(a_p, \alpha_p) \\ (b_1, \beta_1)\ldots(b_{q-1}, \beta_{q-1})(a_1, \alpha_1) \end{matrix}\right) = H^{m,n-1}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix} (a_2, \alpha_2)\ldots(a_p, \alpha_p) \\ (b_1, \beta_1)\ldots(b_{q-1}, \beta_{q-1}) \end{matrix}\right). \tag{C.8}$$

The next important identities needed in the paper are

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1)\ldots(a_p, \alpha_p) \\ (b_1, \beta_1)\ldots(b_q, \beta_q) \end{matrix}\right) = H^{n,m}_{q,p}\left(z\,\frac{1}{z} \,\middle|\, \begin{matrix} (1-b_1, \beta_1)\ldots(1-b_q, \beta_q) \\ (1-a_1, \alpha_1)\ldots(1-a_p, \alpha_p) \end{matrix}\right), \tag{C.9}$$

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1)\ldots(a_p, \alpha_p) \\ (b_1, \beta_1)\ldots(b_q, \beta_q) \end{matrix}\right) = c H^{m,n}_{p,q}\left(z^c \,\middle|\, \begin{matrix} (a_1, c\alpha_1)\ldots(a_p, c\alpha_p) \\ (b_1, c\beta_1)\ldots(b_q, c\beta_q) \end{matrix}\right), \tag{C.10}$$

$$z^{\sigma} H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1)\ldots(a_p, \alpha_p) \\ (b_1, \beta_1)\ldots(b_q, \beta_q) \end{matrix}\right) = H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1+\sigma\alpha_1, \alpha_1)\ldots(a_p+\sigma\alpha_p, \alpha_p) \\ (b_1+\sigma\beta_1, \beta_1)\ldots(b_q+\sigma\beta_q, \beta_q) \end{matrix}\right). \tag{C.11}$$

Many well-known special functions are included in the class of H-functions. For example,

$$\beta^{-1}x^{b/\beta}\exp\{-x^{1/\beta}\} = H_{0,1}^{1,0}\left(x \,\middle|\, \begin{matrix} - \\ (b,\beta) \end{matrix}\right), \tag{C.12}$$

or the Mittag–Leffler function (2.5)

$$E_\beta(-x) = H_{1,2}^{1,1}\left(x \,\middle|\, \begin{matrix} (0,1) \\ (0,1)\,(0,\beta) \end{matrix}\right) = \int_0^\infty e^{-xt} H_{1,1}^{1,0}\left(t \,\middle|\, \begin{matrix} (1-\beta,\beta) \\ (0,1) \end{matrix}\right) dt. \tag{C.13}$$

For the Bessel function of the first kind of order v we have

$$\mathcal{J}_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{v+2k}}{k!\Gamma(k+v+1)} = H_{0,2}^{1,0}\left(\frac{x^2}{4} \,\middle|\, \begin{matrix} \\ (v/2,1)(-v/2,1) \end{matrix}\right), \tag{C.14}$$

while for Wright's generalized Bessel function

$$\mathcal{J}_\lambda^v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^k}{k!\Gamma(1+\lambda+vk)} = H_{0,2}^{1,0}\left(x \,\middle|\, \begin{matrix} - \\ (0,1)(-\lambda,v) \end{matrix}\right). \tag{C.15}$$

From (5.14) of Srivastava *et al.*[72] and (C.14) (see also Prudnikov *et al.*,[61] p. 355, relation 2.25.3.2) we get

$$\begin{aligned} &\int_0^\infty x^{\alpha-1}\mathcal{J}_\lambda^v(\sigma x)\, H_{p,q}^{m,n}\left(\Omega z^r \,\middle|\, \begin{matrix} (a_1,\alpha_1)\ldots(a_p,\alpha_p) \\ (b_1,\beta_1)\ldots(b_q,\beta_1) \end{matrix}\right) dx \\ &\quad = \frac{2^{\alpha-1}}{\sigma^\alpha} H_{p+2,q}^{m,n+1}\left(\Omega\left(\frac{2}{\sigma}\right)^r \,\middle|\, \begin{matrix} (1-\frac{\alpha+r}{2},\frac{r}{2})(a_1,\alpha_1)\ldots(a_p,\alpha_p)(1-\frac{\alpha-v}{2},\frac{r}{2}) \\ (b_1,\beta_1)\ldots(b_q,\beta_q) \end{matrix}\right), \end{aligned} \tag{C.16}$$

where $r, \sigma > 0$; $|\arg\Omega| < \omega\pi/2$, $\omega > 0$ (see (C.2a–C.2h);

$$\operatorname{Re}(\alpha+v) + r \min_{1\leqslant j\leqslant m} \operatorname{Re}(b_j/\beta_j) > 0; \qquad \operatorname{Re}\alpha + r \max_{1\leqslant j\leqslant n} \frac{a_j-1}{\alpha_j} < \frac{3}{2}. \tag{C.17}$$

From (C.13), (C.16) and (C.9) we obtain

$$\int_0^\infty \rho^{n/2} \mathscr{J}_{(n/2)-1}(\rho\,|x|)\, E_\beta(-\mu t^\beta \rho^\gamma)\, d\rho$$

$$= \frac{2^{n/2}}{|x|^{(n+2)/2}} H^{1,2}_{3,2}\left(t^\beta \mu \left(\frac{2}{|x|}\right)^\gamma \,\middle|\, \begin{matrix} (1-\frac{n}{2},\frac{\gamma}{2})(0,1)(0,\frac{\gamma}{2}) \\ (0,1)(0,\beta) \end{matrix} \right)$$

$$= \frac{2^{n/2}}{|x|^{(n+2)/2}} H^{2,1}_{2,3}\left(\frac{|x|^\gamma}{2^\gamma t^\beta \mu} \,\middle|\, \begin{matrix} (1,1)(1,\beta) \\ (n/2,\gamma/2)(1,1)(1,\gamma/2) \end{matrix} \right). \quad \text{(C.18)}$$

From (C.17) we obtain that (C.18) holds for

$$\beta > 0,\ \min(n, \gamma, 2) > (n-1)/2. \quad \text{(C.19)}$$

The latter condition is equivalent to $\gamma = 0$ if $n = 1$; $\gamma > 1/2$ if $n = 2$; $\gamma > 1$ if $n = 3$; $\gamma > 3/2$ if $n = 4$ and $\gamma > 2$ if $n = 5$.

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